

Two-generator elliptic subgroups of  $SL(2, C)$ 

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## Abstract

We classify all two-generator, purely elliptic subgroup of  $SL(2, C)$  by means of the order of generators and the crossing angle of their axes.

## Key words

Möbius transformations, regular polyhedral groups, discrete groups, purely elliptic two-generator subgroup of  $SL(2, C)$ ,

## 1 Introduction

We denote by  $H^3$  the upper-half space and regard it as the 3-dimensional hyperbolic space. The special linear group  $SL(2, C)$  acts in  $H^3$  as a group of hyperbolic isometries. For an elliptic element  $f \in SL(2, C)$ , denote by  $A_f$  the geodesic joining two fixed points of  $f$  on  $\hat{C} = \partial H^3$  and call it the axis of  $f$ . For two elliptic elements  $f, g \in SL(2, C)$ , let  $\langle f, g \rangle$  be the group generated by  $f, g$ . It is well known that  $\langle f, g \rangle$  is purely elliptic if and only if  $A_f \cap A_g \neq \emptyset$ <sup>1)</sup>. In this paper, we classify all two-generator, purely elliptic subgroups of  $SL(2, C)$ .

By  $M(\hat{R}^3)$  we denote the group of Möbius transformations acting on  $\hat{R}^3 = R^3 \cup \{\infty\}$ . The special orthogonal group  $SO(3)$  acts in the unit ball  $B^3$  in  $R^3$  as a rotation group. Let  $\langle f, g \rangle$  be a purely elliptic subgroup of  $SL(2, C)$ . Then there exist  $x_0 \in A_f \cap A_g$  and  $h \in M(\hat{R}^3)$  so that  $h(H^3) = B^3$ ,  $h(x_0) = 0$  and  $h\langle f, g \rangle h^{-1} \subset SO(3)$ . Since  $h(A_f)$  is the geodesic through the origin joining two fixed points of  $hfh^{-1}$ ,  $h(A_f)$  is also called the axis of  $hfh^{-1}$ . Thus, for the study of purely elliptic subgroups, it suffices to investigate subgroups of  $SO(3)$ . From now on, we assume  $f, g \in SO(3)$ . By  $ord(f)$  we denote the order of  $f$ . For non-trivial elements  $f, g$ ,  $\langle f, g \rangle$  may be discrete or not. Suppose that  $\langle f, g \rangle$  is discrete. Then the classification theory of

elementary Kleinian groups yields that  $\langle f, g \rangle$  can be classified into five kinds<sup>3)</sup>: a cyclic group, a dihedral group, a tetrahedral group, a hexahedral group and a dodecahedral group<sup>3)</sup>. The last three kinds are regular polyhedral groups. To describe the non-discrete case, we need some preparations.

For a subgroup  $G$  of  $SO(3)$ , we define

$$F_G = \{ \xi \in S^2 = \partial B^3 \mid g(\xi) = \xi \text{ for some } g \in G - \{id\} \}.$$

For  $k = 2, 3, \dots, +\infty$ , we also define a subset of  $F_G$  as

$$F_G^k = \{ \xi \in F_G \mid g(\xi) = \xi, \text{ ord}(g) = k \}.$$

To describe the action of non-discrete groups on  $S^2$ , we regard  $cl(B^3)$  as the earth. Denote by  $N, S$  and  $E$  the North Pole, the South Pole and the equator, respectively. A non-discrete group  $G$  is said to belong to the class I and denoted by  $G \in C_I$ , if there exists  $h \in SO(3)$  so that  $F_{hGh^{-1}} = h(F_G) = \{N, S\}$ . A group  $G$  belongs to the class II  $C_{II}$  if there exists  $h \in SO(3)$  so that  $cl(h(F_G)) = \{N, S\} \cup E$ ,  $h(F_G) \cap E \subset f(F_G^2)$  and  $h(F_G^k) \cap E = \emptyset$  for  $k \geq 3$ . A group  $G$  belongs to the class III  $C_{III}$  if  $cl(F_G^\infty) = S^2$ . We shall show in Theorem 1 that each non-discrete group in  $SO(3)$  belongs to either  $C_I$ ,  $C_{II}$  or  $C_{III}$ .

For non-trivial elements  $f, g \in SO(3)$ , denote by  $\theta(f, g)$  the crossing angle formed by  $A_f$  and  $A_g$  at the origin. Note that  $\theta(f, g)$  is contained in  $[0, \pi/2]$ . In terms of  $ord(f)$ ,  $ord(g)$  and  $\theta(f, g)$ , we classify

all two-generator subgroups of  $SO(3)$  into eight kinds.

## 2 Non-discrete groups

In the first place, we consider non-discrete groups. As to the classification of subgroups of  $PSL(2C)$ , the following result due to Sullivan is essential.

Proposition 1.<sup>5)</sup> Let  $G$  be a subgroup of  $PSL(2, C)$ . Then, after replacing  $G$  by a subgroup of index 2 if necessary, there are three possibilities:

- (1)  $G$  is discrete.
- (2)  $G$  is solvable and conjugate to a subgroup of similarities.
- (3) (a)  $G$  is dense in  $PSL(2, C)$ , or  
 (b)  $G$  is conjugate to a dense subgroup of  $PSL(2, R)$ , or  
 (c)  $G$  is conjugate to a dense subgroup of  $SO(3)$ .

Suppose that  $G$  is a purely elliptic, non-discrete subgroup of  $SL(2, C)$ . Then the proposition above yields that  $G$  is conjugate in  $M(\hat{R}^3)$  to either (2) a solvable subgroup of similarities or one of its  $Z_2$ -extensions, or (3)(c) a dense subgroup of  $SO(3)$ . Assume that (2) holds. Taking a suitable conjugation in  $SL(2, C)$ , we may suppose that the fixed point set for  $G$  is  $\{0, \infty\}$ . Then  $G$  is isomorphic to a non-discrete subgroup of  $SO(2)$  and hence belongs to  $C_I$ . Next we suppose that  $G$  has a normal subgroup  $G_0$  of index 2, which is isomorphic to a non-discrete subgroup of  $SO(2)$ . Then, for  $f_0 \in G - G_0$ , we have a coset decomposition  $G = G_0 + f_0 G_0$ . The element  $f_0$  is of order 2, so is  $f_0 g$  for each  $g \in G_0$ . Hence  $F_{f_0 G_0}$  forms a dense subset of a great circle on  $S^2$ . Therefore we conclude  $G \in C_{II}$ . Finally we suppose that  $G$  satisfies (3)(c). Taking a suitable conjugation, we may assume that  $G$  is a subgroup of  $SO(3)$  and acts in  $B^3$ . Since  $G$  is dense in  $SO(3)$ , so is  $F_G$  in  $S^2$ . To prove that  $G$  belongs to  $C_{III}$ , it suffices to show  $F_G^\infty \neq \phi$ . So we prove

**Lemma 1.** Let  $G$  be a non-discrete subgroup of  $SO(3)$ . Assume that  $G$  does not leave any great circle on  $S^2$  invariant. Then  $G$  contains an element of infinite order.

**Proof.** We prove this lemma by contradiction.

So we assume that every element in  $G$  is of finite order. Since  $G$  is non-discrete,  $G$  is an infinite group. On the other hand, Selberg's lemma shows that every finitely generated subgroup of  $G$  is finite. As a consequence, we find a finite subset  $X_0$  of  $G$  such that the order of the group  $\langle X_0 \rangle$  is finite and greater than 60. It follows that  $\langle X_0 \rangle$  is conjugate in  $M(\hat{R}^3)$  to an elementary Kleinian group. Therefore we deduce that  $\langle X_0 \rangle$  is a cyclic, a dihedral or a regular polyhedral group. Note that the order of each regular polyhedral group is at most 60<sup>6)</sup>. Hence the group  $\langle X_0 \rangle$  is cyclic or dihedral. In any case, there exists a set  $V$  which consists of two antipodal points on  $S^2$  so that  $V$  is left invariant by  $\langle X_0 \rangle$ . Hence we define a subgroup  $G_0$  of  $G$  by  $G_0 = \{g \in G \mid g(V) = V\}$ . Let  $Y$  be an arbitrary finite subset of  $G$ . As its order  $|\langle X_0 \cup Y \rangle|$  is greater than 60, the group  $\langle X_0 \cup Y \rangle$  is either cyclic or dihedral, too. Since  $V$  is left invariant by  $\langle X_0 \rangle$ , it is also  $\langle X_0 \cup Y \rangle$ -invariant. It implies that  $\langle X_0 \cup Y \rangle$  is a subgroup of  $G_0$  and hence  $G_0$  contains every finitely generated subgroup of  $G$ . It means  $G = G_0$  and  $G$  leaves a great circle on  $S^2$  invariant. It contradicts the assumption and our lemma is established.

We summarize the argument above as the following form.

**Theorem 1.** Let  $G$  be a non-discrete subgroup of  $SO(3)$ . Then  $G$  belongs to either  $C_I$ ,  $C_{II}$  or  $C_{III}$ .

Let  $G$  be a group generated by  $f, g \in SO(3)$ . From now on, we add the condition  $ord(f) \geq ord(g)$  in the notation  $G = \langle f, g \rangle$ . Obviously we have the following equivalences:

$$\begin{aligned} G = \langle f, g \rangle \in C_I &\Leftrightarrow \theta(f, g) = 0 \text{ and } ord(f) = \infty. \\ G = \langle f, g \rangle \in C_{II} &\Leftrightarrow (1) \theta(f, g) = \pi/2 \text{ and } ord(f) = \infty, \\ &\quad ord(g) = 2 \text{ or} \\ &\quad (2) \theta(f, g)/\pi \text{ is irrational and} \\ &\quad ord(f) = ord(g) = 2. \end{aligned}$$

As an immediate consequence of Theorem 1, we have the following which is closely related to Jørgensen's result<sup>4)</sup>.

**Corollary 1.** Let  $G$  be a purely elliptic subgroup of  $SL(2, C)$ . Assume that  $G$  does not leave any circle in  $C$  invariant. Then  $G$  is discrete if and only if each cyclic subgroup of  $G$  is discrete.

If the assumption of this corollary is not satisfied, we can construct a counter example as follows: Let  $G$  be a group consisting of all rotations of finite order which fix  $0, \infty$ . Of course  $G$  is non-discrete, but every finitely generated subgroup is discrete.

### 3 Regular polyhedral groups

Each discrete subgroup of  $SO(3)$  is a cyclic, a dihedral or a regular polyhedral group. So, to consider a discrete group  $\langle f, g \rangle$ , we may restrict  $f, g$  to elements in one of these groups. It suffices to investigate only regular polyhedral groups. First, we deal with the tetrahedral groups.

[I] **The tetrahedral group.** A tetrahedral group is generated by rotations  $f_1, \dots, f_4$  of order 3, and  $g_1, g_2, g_3$  of order 2. Each  $f_i$  is a rotation by  $2\pi/3$  about the axis through a vertex and the center of the opposite face. Each  $g_i$  is a rotation by  $\pi$  about the axis joining two midpoints of opposite edges. For  $\{f, g\} \subset \langle f_1, \dots, g_3 \rangle$ , only two cases can occur for  $\langle f, g \rangle$  to be a tetrahedral group.

- (1)  $\theta(f, g) = \arctan 2\sqrt{2} = 70.5287 \dots^\circ$   
and  $\text{ord}(f) = \text{ord}(g) = 3$ .
- (2)  $\theta(f, g) = \arctan \sqrt{2} = 54.7376 \dots^\circ$   
and  $\text{ord}(f) = 3, \text{ord}(g) = 2$ .

In any other combination,  $\langle f, g \rangle$  is a cyclic or a dihedral group.

[II] **The hexahedral group.** A hexahedral group is generated by three kinds of rotations: rotations  $f_1, f_2, f_3$  of order 4,  $g_1, \dots, g_4$  of order 3 and  $h_1, \dots, h_6$  of order 2. Each  $f_i, g_i$  and  $h_i$  is a rotation by  $\pi/2, 2\pi/3$  and  $\pi$  about the axis through two centers of opposite faces, two opposite vertices and two midpoints of opposite edges, respectively. We find six combinations of  $\{f, g\} \subset \langle f_1, \dots, h_6 \rangle$  for  $\langle f, g \rangle$  to be a regular polyhedral group. In (1), ..., (4)  $\langle f, g \rangle$  is a hexahedral group and a tetrahedral group in (5), (6).

- (1)  $\theta(f, g) = \pi/2$  and  $\text{ord}(f) = \text{ord}(g) = 4$ .
- (2)  $\theta(f, g) = \arctan \sqrt{2} = 54.7376 \dots^\circ$   
and  $\text{ord}(f) = 4, \text{ord}(g) = 3$ .
- (3)  $\theta(f, g) = \pi/4$  and  $\text{ord}(f) = 4, \text{ord}(g) = 2$ .
- (4)  $\theta(f, g) = \arctan (1/\sqrt{2}) = 35.2643 \dots^\circ$   
and  $\text{ord}(f) = 3, \text{ord}(g) = 2$ .
- (5)  $\theta(f, g) = \arctan 2\sqrt{2} = 70.5287 \dots^\circ$   
and  $\text{ord}(f) = \text{ord}(g) = 3$ .

- (6)  $\theta(f, g) = \arctan \sqrt{2} = 54.7376 \dots^\circ$   
and  $\text{ord}(f) = 3, \text{ord}(g) = 2$ .

[III] **The dodecahedral group.** Generators of a dodecahedral group are rotations  $f_1, \dots, f_6$  of order 5,  $g_1, \dots, g_{10}$  of order 3 and  $h_1, \dots, h_{15}$  of order 2. Each  $f_i, g_i$  and  $h_i$  is a rotation by  $2\pi/5, 2\pi/3$  and  $\pi$  about the axis through two centers of opposite faces, two opposite vertices and two midpoints of opposite edges, respectively. There are ten combinations for  $\langle f, g \rangle$  to be a regular polyhedral group. Each group in (1), ..., (8) corresponds to a dodecahedral group, and a tetrahedral group in (9), (10).

- (1)  $\theta(f, g) = \arctan 2 = 63.4349 \dots^\circ$   
and  $\text{ord}(f) = \text{ord}(g) = 5$ .
- (2)  $\theta(f, g) = \arctan (3 - \sqrt{5}) = 37.3773 \dots^\circ$   
and  $\text{ord}(f) = 5, \text{ord}(g) = 3$ .
- (3)  $\theta(f, g) = \arctan (3 + \sqrt{5}) = 79.1876 \dots^\circ$   
and  $\text{ord}(f) = 5, \text{ord}(g) = 3$ .
- (4)  $\theta(f, g) = \arctan \{(\sqrt{5} - 1)/2\} = 31.7174 \dots^\circ$   
and  $\text{ord}(f) = 5, \text{ord}(g) = 2$ .
- (5)  $\theta(f, g) = \arctan \{(\sqrt{5} + 1)/2\} = 58.2825 \dots^\circ$   
and  $\text{ord}(f) = 5, \text{ord}(g) = 2$ .
- (6)  $\theta(f, g) = \arctan (2/\sqrt{5}) = 41.8103 \dots^\circ$   
and  $\text{ord}(f) = \text{ord}(g) = 3$ .
- (7)  $\theta(f, g) = \arctan \{(3 - \sqrt{5})/2\} = 20.9051 \dots^\circ$   
and  $\text{ord}(f) = 3, \text{ord}(g) = 2$ .
- (8)  $\theta(f, g) = \arctan \{(3 + \sqrt{5})/2\} = 69.0948 \dots^\circ$   
and  $\text{ord}(f) = 3, \text{ord}(g) = 2$ .
- (9)  $\theta(f, g) = \arctan 2\sqrt{2} = 70.5287 \dots^\circ$   
and  $\text{ord}(f) = \text{ord}(g) = 3$ .
- (10)  $\theta(f, g) = \arctan \sqrt{2} = 54.7356 \dots^\circ$   
and  $\text{ord}(f) = 3, \text{ord}(g) = 2$ .

In any other case,  $\langle f, g \rangle$  is a cyclic or a dihedral group.

By removing duplications, we have seventeen combinations for  $\langle f, g \rangle$  to be a discrete group. In any other combination,  $\langle f, g \rangle$  is not discrete.

### 4 The dodecahedral group

Cases of the tetrahedral and the hexahedral groups are quite simple, so we discuss only the dodecahedral groups. Let  $P_1, \dots, P_{20}$  and  $F_1, \dots, F_{12}$  be vertices and faces of the regular dodecahedron  $\Delta$ , respectively (See Figures 1 and 2). For each face  $F_k, k = 1, \dots, 6$  of  $\Delta$ , there exists the opposite

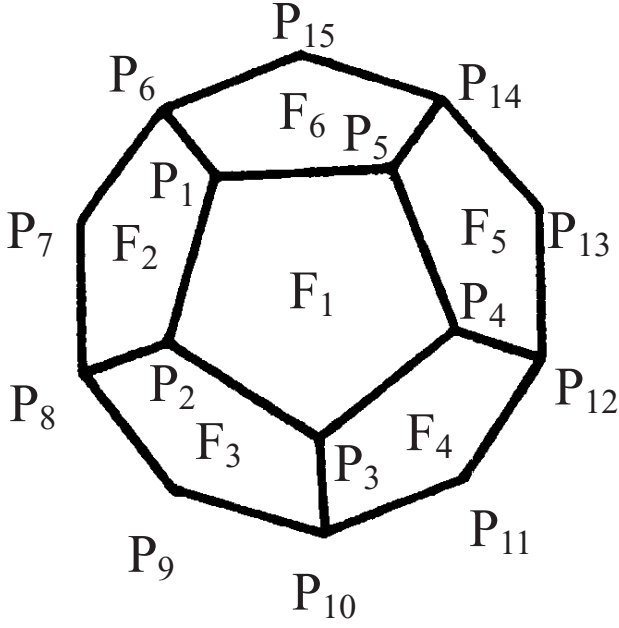


Figure 1

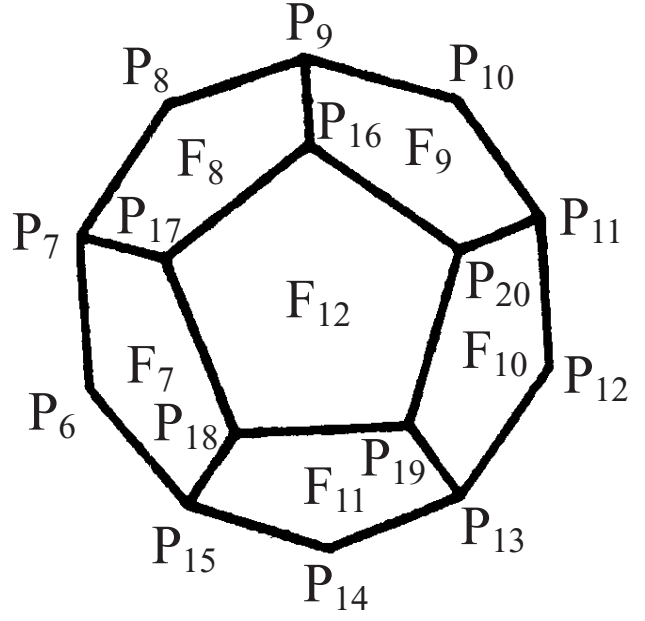


Figure 2

face  $F_l$ ,  $l=7, \dots, 12$ . Denote by  $f_k$  the rotation by  $2\pi/5$  about the line through the centers of  $F_k$  and  $F_l$ . This rotation  $f_k$  is oriented so as to interchange vertices of  $F_k$  clockwise. Similarly, for each  $P_k$ ,  $k=1, \dots, 10$  we can find the opposite vertex  $P_l$ ,  $l=11, \dots, 20$ . Let  $g_k$  be the rotation by  $2\pi/3$  about the line through  $P_k$  and  $P_l$ . So as to interchange three vertices around  $P_k$  clockwise,  $g_k$  is oriented. Let  $h_1$  be the rotation by  $\pi$  about the line through the midpoints of edges  $P_1P_2$  and  $P_{20}P_{19}$ . We denote it by  $h_1: (1,2) - (20,19)$ . In this way, we can enumerate all rotations of order 2 as the following:  $h_2: (2,3) - (19,18)$ ,  $h_3: (3,4) - (18,17)$ ,  $h_4: (4,5) - (17,16)$ ,  $h_5: (5,1) - (16,20)$ ,  $h_6: (1,6) - (20,11)$ ,  $h_7: (2,8) - (19,13)$ ,  $h_8: (3,10) - (18,15)$ ,  $h_9: (4,12) - (17,7)$ ,  $h_{10}: (5,14) - (16,9)$ ,  $h_{11}: (6,7) - (11,12)$ ,  $h_{12}: (7,8) - (12,13)$ ,  $h_{13}: (8,9) - (13,14)$ ,  $h_{14}: (9,10) - (14,15)$ ,  $h_{15}: (10,11) - (15,6)$ .

Now we denote by  $D_k$ ,  $T$  and  $D$  the dihedral group of degree  $k$ , the tetrahedral and the dodecahedral group, respectively. The group  $D$  is generated by  $S = \{f_1, \dots, f_6, g_1, \dots, g_{10}, h_1, \dots, h_{15}\}$ . We consider all two-generator subgroups of  $D$ . Since the order of each element in  $S$  is prime, it suffices to consider only groups that are generated by two elements in  $S$ . We may exclude cyclic subgroups of  $D$ .

Subgroups of  $D$  generated by two elements in  $S$  are classified into six types as the followings: (1)

$\langle f_i, f_j \rangle$ , (2)  $\langle g_i, g_j \rangle$  (3)  $\langle h_i, h_j \rangle$  (4)  $\langle f_i, g_j \rangle$  (5)  $\langle f_i, h_j \rangle$  and (6)  $\langle g_i, h_j \rangle$ . In any case, it suffices to show when  $i=1$

(1)  $\langle f_i, f_j \rangle$ ,  $i, j=1, \dots, 6$  and  $i \neq j$ .

We may assume  $j=2$ . As  $f_1 f_2(P_k) = P_k$  ( $k=1, 20$ ) and  $f_1 f_2$  transposes  $P_5, P_2$  and  $P_6$  cyclically, we deduce  $f_1 f_2 = g_1$ . Similarly,  $f_2 f_1(P_k) = P_k$  ( $k=2, 19$ ) and  $f_2 f_1$  transposes  $P_1, P_3$  and  $P_8$  cyclically, so we have  $f_2 f_1 = g_2$ . In this way, we obtain  $g_k \in \langle f_1, f_2 \rangle$  for  $k=1, \dots, 10$ . On the other hand  $f_1 g_2$  interchanges  $P_1$  and  $P_2$ , so we have  $f_1 g_2 = h_1$ . Therefore we conclude  $h_k \in \langle f_1, f_2 \rangle$  for  $k=1, \dots, 15$  and hence  $\langle f_1, f_2 \rangle = D$ . Let  $\xi$  be the dihedral angle of  $\Delta$  and set  $\vartheta = \theta(f_1, f_2)$ . Since  $\sin \xi = 2/\sqrt{5}$ ,  $\xi > \pi/2$  and  $\xi + \vartheta = \pi$ , we have  $\vartheta = \arctan 2$ .

(2)  $\langle g_i, g_j \rangle$ ,  $i, j=1, \dots, 10$  and  $i \neq j$ .

We discuss these groups by classifying them into two cases.

(i) The case where fixed vertices of  $g_1$  and  $g_j$  belong to a common edge, i.e.  $j=2, 5, 6$ . Without loss of generalities, we may set  $j=2$ . As  $g_2 g_1(F_1) = F_1$  and  $g_2 g_1(P_1) = P_3$ , it follows  $g_2 g_1 = f_1^3$  and  $f_1 \in \langle g_1, g_2 \rangle$ . Noting  $g_1 f_1 g_1^{-1} = f_2$ , we conclude  $\langle g_1, g_2 \rangle \supset \langle f_1, f_2 \rangle = D$ . Denote by  $a$  the length of edges of  $\Delta$ . We set  $\vartheta = \theta(g_1, g_2)$ . Since the radius of the circumscribed sphere of  $\Delta$  is  $\sqrt{3}(\sqrt{5}+1)a/4^{2/3}$ , we obtain  $\cos \vartheta = \sqrt{5}/3$  and so  $\vartheta = \arctan(2/\sqrt{5})$ .

(ii) The case where fixed vertices of  $g_1, g_j$  do not

belong to any common edge, i.e.  $j=3,4,7,8,9,10$ . We may assume  $j=3$  and denote  $\vartheta = \theta(g_1, g_3)$ .

Since the length of the edge  $P_1P_3$  is  $(\sqrt{5}+1)a/2$ , the second cosine formula yields  $\vartheta = \arctan 2\sqrt{2}$ . In general, two generators of order 3 whose axes cross at the angle  $\arctan 2\sqrt{2}$  generate the tetrahedral group  $T$ . So we conclude  $\langle g_1, g_3 \rangle = T$ .

(3)  $\langle h_i, h_j \rangle$ ,  $i, j=1, \dots, 15$  and  $i \neq j$ .

Since both generators are of order two, it follows that  $\langle h_i, h_j \rangle$  is  $D_k$  for each  $i, j, i \neq j$ . We divide this case into three parts.

(i)  $j=2, 5, 6, 7, 8, 10, 13, 15$ . In this case we know  $\vartheta = \theta(h_1, h_j) = 2\pi/5$ ,  $\langle h_i, h_j \rangle = D_5$ .

(ii)  $j=3, 4, 11, 12$ . In this case we see  $\vartheta = \pi/3$  and obtain  $\langle h_1, h_j \rangle = D_3$ .

(iii) If  $j=9, 14$ , we deduce  $\vartheta = \pi/2$  and obtain  $\langle h_1, h_j \rangle = D_2$ .

(4)  $\langle f_i, g_j \rangle$ ,  $i=1, \dots, 6$  and  $j=1, \dots, 10$ .

Any  $g_j$  ( $j=1, \dots, 10$ ) does not leave the axis of  $f_1$  invariant. Thus, for each  $j$ , there exists  $k$  ( $\neq 1$ ) so that  $g_j f_1 g_j^{-1} = f_k$ . Therefore we conclude  $\langle f_1, g_j \rangle \supset \langle f_1, f_k \rangle = D$ . To calculate  $\vartheta = \theta(f_1, g_j)$ , we consider two possibilities:

(i) The case where  $F_1$  contains a fixed vertex of  $g_j$ , that is  $j=1, \dots, 5$ .

(ii) The case where  $F_1$  contains no fixed vertices of  $g_j$ , i.e.  $j=6, \dots, 10$ .

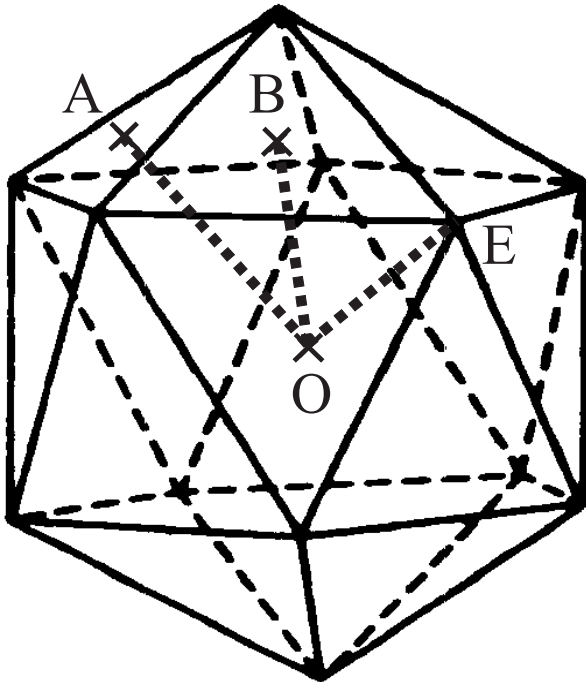


Figure 3

In both cases, we investigate the regular icosahedron  $\Lambda$  which is the dual polyhedron of  $\Delta$ . We denote by  $E$  the vertex of  $\Lambda$  which corresponds to the center of the face  $F_1$  of  $\Delta$ . Centers of faces of  $\Lambda$  which correspond to vertices  $P_1, P_6$  of  $\Delta$  are denoted by  $B, A$  respectively (See Figure 3). Let  $O$  be the center of the circumscribed sphere of  $\Lambda$  ( $\Delta$ ). By setting  $\angle BOA = \vartheta_1$ ,  $\angle EOB = \vartheta_2$ , we have  $\theta(f_1, g_1) = \vartheta_2$  and  $\theta(f_1, g_6) = \vartheta_1 + \vartheta_2$ . Let  $\eta$  be the dihedral angle of  $\Lambda$ . Since  $\sin \eta = 2/3$  and  $\eta + \vartheta_1 = \pi$ , we see  $\tan \vartheta_1 = 2/\sqrt{5}$ . We denote by  $b$  the length of edges of  $\Lambda$ . It is well known that the radii of the inscribed and circumscribed spheres of  $\Lambda$  are  $(3+\sqrt{5})b/4\sqrt{3}$  and  $\sqrt{5+\sqrt{5}}b/2\sqrt{2}$ , respectively<sup>29</sup>. By observing the triangle  $\triangle OBE$ , we have  $\tan \vartheta_2 = 3-\sqrt{5}$  and so  $\tan(\vartheta_1 + \vartheta_2) = 3+\sqrt{5}$ . Therefore we obtain  $\theta(f_1, g_1) = \arctan(3-\sqrt{5})$  and  $\theta(f_1, g_6) = \arctan(3+\sqrt{5})$ .

(5)  $\langle f_i, h_j \rangle$ ,  $i=1, \dots, 6$  and  $j=1, \dots, 15$ .

We classify these groups into three cases.

(i) The case where  $F_1$  contains an invariant edge of  $h_j$ , i.e.  $j=1, \dots, 5$ .

Since  $h_j f_1 h_j^{-1} = f_{1+j}$  for each  $j$ , it implies  $\langle f_1, h_j \rangle \supset \langle f_1, f_{1+j} \rangle = D$ . We set  $\theta(f_1, h_1) = \vartheta$ . As  $2\vartheta + \xi = \pi$  with  $\sin \xi = 2/\sqrt{5}$ ,  $\xi > \pi/2$ , we see  $\cos 2\vartheta = 1/\sqrt{5}$ . Therefore we conclude  $\theta(f_1, h_1) = \arctan\{(\sqrt{5}-1)/2\}$ .

(ii) The case where  $F_1$  does not contain invariant edges of  $h_j$  and one of them contains a vertex of  $F_1$ , that is  $j=6, \dots, 10$ . We may assume  $j=6$ . Since  $h_6 f_1 h_6^{-1} = f_4^{-1}$ , it follows  $\langle f_1, h_6 \rangle \supset \langle f_1, f_4 \rangle = D$ . We have  $2\theta(f_1, h_6) + \theta(f_1, h_4) = \pi$  and  $\theta(f_1, h_4) = \arctan 2$  from the case (i). So we obtain  $\tan 2\theta(f_1, h_6) = -2$  and hence  $\theta(f_1, h_6) = \arctan\{(\sqrt{5}+1)/2\}$ .

(iii) The case where the axes of  $f_1$  and  $h_j$  cross at right angles, i.e.  $j=11, \dots, 15$ .

In this case we can easily see  $\theta(f_1, h_j) = \pi/2$  and hence  $\langle f_1, h_j \rangle = D_5$ .

(6)  $\langle g_i, h_j \rangle$ ,  $i=1, \dots, 10$  and  $j=1, \dots, 15$ .

This case is divided into four parts.

(i) The case where  $P_1$  belongs to an invariant edge  $E_j$  of  $h_j$ , i.e.  $j=1, 5, 6$ .

We deal only with  $\langle g_1, h_1 \rangle$ . Noting  $h_1 g_1(F_1) = F_1$  and  $h_1 g_1(P_1) = P_2$ , we have  $h_1 g_1 = f_1^{-1}$  and hence  $f_1 \in \langle g_1, h_1 \rangle$ . Since  $h_1 f_1 h_1^{-1} = f_2$ , it implies  $\langle g_1, h_1 \rangle \supset \langle f_1, f_2 \rangle = D$ . We define  $\theta(g_1, h_1) = \vartheta$ . Denote the length of the edge  $P_1P_2$  by  $a$ . Since  $r$ , the length of



$OP_1$ , is the radius of the circumscribed sphere of  $\Delta$ ,  $r = \sqrt{3}(\sqrt{5}+1)\alpha/4$  and hence  $\cos \vartheta = \sqrt{3}(\sqrt{5}+1)/6$ . Therefore we obtain  $\vartheta = \arctan\{(3-\sqrt{5})/2\}$ .

(ii) The case where  $P_1$  and  $E_j$  belong to a common face and  $P_1$  is opposite to  $E_j$  in this face, that is  $j = 3, 12, 14$ . We may suppose  $j = 3$ . As  $g_1^{-1}h_3(F_5) = F_5$ ,  $g_1^{-1}h_3(P_4) = P_{14}$ , it follows  $g_1^{-1}h_3 = f_5^2$  and  $f_5 \in \langle g_1, h_3 \rangle$ . Since  $h_3 f_5 h_3^{-1} = f_3$ , we know  $\langle g_1, h_3 \rangle \supset \langle f_3, f_5 \rangle = D$ . Denote  $\theta(g_1, h_3) = \vartheta$ . Then we can see  $2\vartheta + \theta(g_1, g_6) = 2\vartheta + \arctan(2/\sqrt{5}) = \pi$ . So we obtain  $\vartheta = \arctan\{(3+\sqrt{5})/2\}$ .

(iii) The case where  $P_1$  adjoins an invariant edge  $E_j$  of  $h_j$  i. e.  $j = 2, 4, 7, 10, 11, 15$ . We set  $j = 2$ . As  $h_2 g_1(P_4) = P_4$ ,  $h_2 g_1(P_3) = P_{12}$ , it implies  $h_2 g_1 = g_4^{-1}$  and  $\langle g_1, h_2 \rangle = \langle g_1, g_4 \rangle$ . Obviously we have  $\theta(g_1, g_4) = \theta(g_1, g_3)$ . Therefore, from (2)(ii), we obtain  $\langle g_1, h_2 \rangle = \langle g_1, g_4 \rangle = T$ , a tetrahedral group, and hence  $\theta(g_1, h_2) = \arctan 2\sqrt{2}$ .

(iv) The case where  $P_1$  and  $E_j$  do not belong to any common face, that is  $j = 8, 9, 13$ . In this case we can easily obtain  $\theta(g_1, h_j) = \pi/2$  and  $\langle g_1, h_j \rangle = D_3$ .

Thus we complete the proof of the dodecahedral case.

## 5. Two-generator groups

Let us state our main result.

**Theorem 2.** Let  $f, g$  be non-trivial elements in  $SO(3)$ . Then the following holds:

### [I] discrete case

- (1) If  $\theta(f, g) = 0$  and  $\text{ord}(f) < \infty$ , then  $\langle f, g \rangle$  is a finite cyclic group.
- (2) (a) If  $\theta(f, g) = \pi/2$  and  $\text{ord}(f) < \infty$ ,  $\text{ord}(g) = 2$ , or  
(b) If  $\theta(f, g)/\pi$  is non-zero rational and  $\text{ord}(f) = \text{ord}(g) = 2$ , then  $\langle f, g \rangle$  is a dihedral group.
- (3) (a) If  $\theta(f, g) = \arctan 2\sqrt{2}$  and  $\text{ord}(f) = \text{ord}(g) = 3$ , or  
(b)  $\theta(f, g) = \arctan \sqrt{2}$  and  $\text{ord}(f) = 3$ ,  $\text{ord}(g) = 2$ , then  $\langle f, g \rangle$  is a tetrahedral group.
- (4) (a) If  $\theta(f, g) = \pi/2$  and  $\text{ord}(f) = \text{ord}(g) = 4$ , or  
(b) If  $\theta(f, g) = \arctan \sqrt{2}$  and  $\text{ord}(f) = 4$ ,  $\text{ord}(g) = 3$ , or  
(c) If  $\theta(f, g) = \pi/4$  and  $\text{ord}(f) = 4$ ,  $\text{ord}(g) = 2$ , or  
(d) If  $\theta(f, g) = \arctan(1/\sqrt{2})$  and  $\text{ord}(f) = 3$ ,  $\text{ord}(g) = 2$ , then  $\langle f, g \rangle$  is a hexahedral group.

- (5) (a) If  $\theta(f, g) = \arctan 2$  and  $\text{ord}(f) = \text{ord}(g) = 5$ , or  
(b) If  $\theta(f, g) = \arctan(3-\sqrt{5})$  and  $\text{ord}(f) = 5$ ,  $\text{ord}(g) = 3$ , or  
(c) If  $\theta(f, g) = \arctan(3+\sqrt{5})$  and  $\text{ord}(f) = 5$ ,  $\text{ord}(g) = 3$ , or  
(d) If  $\theta(f, g) = \arctan\{(\sqrt{5}-1)/2\}$  and  $\text{ord}(f) = 5$ ,  $\text{ord}(g) = 2$ , or  
(e) If  $\theta(f, g) = \arctan\{(\sqrt{5}+1)/2\}$  and  $\text{ord}(f) = 5$ ,  $\text{ord}(g) = 2$ , or  
(f) If  $\theta(f, g) = \arctan(2/\sqrt{5})$  and  $\text{ord}(f) = \text{ord}(g) = 3$ , or  
(g) If  $\theta(f, g) = \arctan\{(3-\sqrt{5})/2\}$  and  $\text{ord}(f) = 3$ ,  $\text{ord}(g) = 2$ , or  
(h) If  $\theta(f, g) = \arctan\{(3+\sqrt{5})/2\}$  and  $\text{ord}(f) = 3$ ,  $\text{ord}(g) = 2$ , then  $\langle f, g \rangle$  is a dodecahedral group.

### [II] non-discrete case

- (1) If  $\theta(f, g) = 0$  and  $\text{ord}(f) = \infty$ , then  $\langle f, g \rangle$  belongs to  $C_I$ .
- (2) (a) If  $\theta(f, g) = \pi/2$  and  $\text{ord}(f) = \infty$ ,  $\text{ord}(g) = 2$ , or  
(b) If  $\theta(f, g)/\pi$  is irrational and  $\text{ord}(f) = \text{ord}(g) = 2$ , then  $\langle f, g \rangle$  belongs to  $C_{II}$ .
- (3) In any other than [I](1), ..., [II](2),  $\langle f, g \rangle$  belongs to  $C_{III}$ .

As a consequence of this theorem, we give a sufficient condition for  $\langle f, g \rangle$  to be a group in  $C_{III}$ . Let  $f, g$  be non-trivial elements of  $SO(3)$ . If  $\theta(f, g) \neq 0$ , then  $\langle f, g \rangle$  cannot only be a cyclic group but also a group in  $C_I$ . On the other hand, it is necessary that  $\theta(f, g) = \pi/2$  or  $\text{ord}(f) = \text{ord}(g) = 2$  for  $\langle f, g \rangle$  to be a dihedral group in  $C_{II}$ . Moreover, orders of elements in each regular polyhedral groups are at most  $5^{(3,5)}$ . Hence we obtain

**Corollary 2** Let  $f, g$  be non-trivial elements in  $SO(3)$ . If  $\theta(f, g)$  is contained in  $(0, \pi/2)$  and  $\text{ord}(f) > 5$ , then  $\langle f, g \rangle$  belongs to  $C_{III}$ .

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## $SL(2, C)$ の 2 元生成、純楕円的部分群

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### 要 旨

$SL(2, C)$  の純楕円的部分群は、離散群としては有限巡回群、2 面体群、正 4 面体群、正 6 面体群、正 12 面体群からなり、非離散部分群は 3 つのクラス  $C_I$ 、 $C_{II}$ 、 $C_{III}$  に分類される。本論分は  $SL(2, C)$  のすべての 2 元生成純楕円的部分群を、その生成元の位数および軸の交角により完全に分類した。